

1 Problem Statement

In this version of Colored Bin-Packing, we have n items of unit weight, where each item is one of x colors, where $x \geq 3$. Again, we have an unlimited supply of bins, each with weight limit L , in which to pack the items and our goal is to minimize the total number of bins.

2 Algorithm

The unit-weight case introduces weight constraints. Furthermore, we know that color constraints, specifically the discrepancy, forces us to use more bins than the weight constraint alone. The first half of the algorithm depends on discrepancy. If discrepancy is not a problem (i.e. is not more than 0), we simply order and split the bins up similar to the zero-weight case. If discrepancy is an issue (i.e. is more than 0), we handle the bins differently depending on if bin capacity is even or odd. In both cases we pack bins by alternating between items of the most frequent color and items of other colors. If bin capacity is even, we may be able to condense bins by using items of other colors that top some bins to merge bins containing only a single item of the most frequent color. If bin capacity is odd, we check if we can eventually reduce discrepancy to zero. This is because we use one additional max-color item for every bin we pack. If discrepancy eventually becomes zero, we are in the case where discrepancy is not a problem, and apply that algorithm. If this is not so, we will end up with bins topped by items of the most frequent color. The final set will be optimal, since all the bins will be topped with items of the this color and therefore cannot be combined.

The UNIT-WEIGHT algorithm solves this problem.

We first note that we must use at least $\left\lceil \frac{n}{L} \right\rceil$ number of bins.

Let D denote the discrepancy, or the difference between the number of *MaxColor* items and *OtherColors* items. By the previous proof, we know that for 0-weight items, we will either need one bin (in the case where $D \leq 0$), or D bins (in the case where $D > 0$).

3 Proof

Case 1: $D \leq 0$ (first if statement) If the discrepancy is not more than zero, we use the ZERO-WEIGHT algorithm to first order the items so they satisfy the color requirements. Then, without breaking this order, we pack the first L items in the first bin, the next L items in the second, and so on, until we have packed all the items.

Example:

Items: 4 W, 3 B, 2 Y

Weight Limit: 3

In the zero-weight case, we would sort these items in this order, using only one bin:

BWBWBWYWY

We use this result to get the unit weight solution:

BWB WBW YWY

where a space denotes a new bin.

This is based on the two requirements defined by the problem: color and weight. We know from the zero-weight case that the ZERO-WEIGHT algorithm satisfies the color restriction when discrepancy is less than or equal to 0. By allocating the items according to the weight limit, we use the minimum amount of bins necessary based on bin capacity. Since both are satisfied, the solution is optimal.

Case 2: $D > 0$ (else statement)

(a) When L is even.

Example:

Items: 12 W, 3 B, 2 Y 2 G

Weight Limit: 4

D: $12 - (3 + 2 + 2) = 12 - 7 = 5$

We use UNITWEIGHT to first pack these into bins,

WBWB WYWG WBWY WGW W W W W

where a space denotes a new bin. Note that there are four bins that only contains a single *MAXCOLOR* - white - item. We reduce the number of bins further by calling CONDENSE. This becomes,

WBW WYW WBWY WGW WBW WGW

giving us 6 bins instead of 8.

Note that we use the same definition of N-bins, H-bins and F-bins seen in CONDENSE.

With a positive discrepancy, it is easy to see that the color requirement will force us to have more bins than the minimum number of bins enforced by the weight limit. This is because each excess *MaxColors* item requires an individual bin - these bins form the set of 'M-bins'.

Keeping this in mind, we also want to use *OtherColors* items to buffer as many *MaxColor* items as possible. We do this to minimize the effect excess *MaxColor* items will have on the eventual number of bins. The more *MaxColor* items we buffer, the fewer M-bins we are forced to create. Hence we start every bin with a *MaxColor* item, and place only one *OtherColors* item between each *MaxColor* item. Completely packed bins topped with an *OtherColors* item form the set called 'F-bins'.

When we run out of *OtherColors* items while packing a bin, we get a potential P-bins. To use up as many *MaxColor* items as possible, we cover the *OtherColors* item currently topping the bin, with a *MaxColor* item. If the bin is able to accept 2 or more items after we do so, we get a P-bins. Because this happens only when we run out of *OtherColors* items, a P-bins only occurs once.

We refer to the packing before the call to CONDENSE as the *initial packing*. After the initial packing, S contains F-bins, M-bins and a possible P-bins. Because L is even and all F-bins start with a *MaxColor* item, all F-bins must be topped with an *OtherColors* item. These *OtherColors* items are used to combine the M-bins, and further reduce our total number of bins using the algorithm CONDENSE.

To show CONDENSE's optimality, we prove that the number of bins in S cannot be further reduced after CONDENSE is run.

There are several cases, all of which are dependent on CONDENSE's exit condition.

- i. S contains no F-bins.
- ii. S contains no M-bins.
- iii. S contains neither F-bins nor M-bins.

Since combining bins requires both F-bins and M-bins, any one of these three conditions means that we cannot reduce the total number of bins by combining any bins in S .

(b) When L is odd.

We observe that when the weight limit is odd, if we use as many *MaxColor* items as possible when packing a bin, we use one more *MaxColor* than *OtherColors* item for every bin. To do so, we fill the bin in a similar fashion to if L is even - starting with a *MaxColor* item and alternating with *OtherColors* items. However, because the L is odd, we end with a *MaxColor* item. This is the excess *MaxColor* item we pack when packing each bin.

Next, we note that if $D > 0$, the number of M-bins depends on the number of *OtherColors* items. This is because since we alternate between *OtherColors* and *MaxColor* items when packing the bins, each bin takes exactly $\lfloor L/2 \rfloor$ *OtherColors* items. If *OtherCount* is the total number of *OtherColors*, we can form $\lceil \frac{\text{OtherCount}}{\lfloor L/2 \rfloor} \rceil$ P-bins and F-bins. The remaining *MaxColor* items must each be packed in a separate bin which are the M-bins.

Thus, if we can form enough P-bins and F-bins to pack the excess *MaxColor* items, we can reduce D to 0, otherwise we form M-bins. This gives us 2 cases.

(a) We eventually reduce D to 0.

Example:

Items: 8 W, 3 B, 2 Y 2 G

Weight Limit: 5

$$\underline{D}: 8 - (3 + 2 + 2) = 8 - 7 = 1$$

We use UNITWEIGHT to first pack these into bins.

After the first bin,

WBWBW

where a space denotes a new bin, DISCREPANCY is now zero and we pack the remaining items using SPLIT. We first sort the remaining items,

WYWGBWBWYG

before splitting the sorted items into bins to give us,

WYWGW BWYG

This gives us,

WBWBW WYWGW BWYG

as our overall solution.

If $D \leq \lceil \frac{OtherCount}{\lfloor L/2 \rfloor} \rceil$, we know that $D = 0$ after D bins are filled in Step – (fill in). From Case 1, we know that SPLIT gives us an optimal packing when $D = 0$. Thus, for S to be optimal, the D bins packed during this phase must also be optimal.

To show that the D bins are optimally packed, we note that the initial packing falls into two categories:

- i. Initial packing yields a P-bin:

If the initial packing yields a P-bin, it means that $D = 1$ when we started filling the P-bin. The phase ends when we place the last excess *MaxColor* item into the P-bin. This reduces D to 0.

In this case, we show that there are no more items left.

Since $D = 0$, we know the number of *MaxColor* and *OtherColors* items must be equal. Any additional *OtherColors* item would mean the P-bins would not exist, as the last item placed into the bin would be an *OtherColors* item. This is a contradiction. Since there are no *OtherColors* items remaining there must be no *MaxColor* items remaining. Since there are no items remaining, we cannot combine any of the bins, and the solution set is optimal.

- ii. Initial packing does not yield any P-bins.

Note that we will not have any M-bins. When $D = 0$, there are two cases:

- A. $MaxCount = OtherCount = 0$:
We won't need any additional bins so clearly there are no M-bins.
- B. $MaxCount = OtherCount > 0$: Since there are no excess $MaxColor$ items, there will be no bins containing only a $MaxColor$ item.
Here, the solution set contains only F-bins. Since we have no M-bins, bin combination cannot happen, and the set is optimal.

- iii. D remains greater than 0.

Example:

Items: 15 W, 3 B, 2 Y 2 G

Weight Limit: 5

D: $15 - (3 + 2 + 2) = 15 - 7 = 6$

We use UNITWEIGHT to pack these into bins, knowing that D will not eventually reduce to zero.

WBWBW WYWGW WBWYW WG W W W W

where a space denotes a new bin. Though there are bins containing only one items, we cannot condense them since all the non- $MAXCOLOR$ items need them in their bins. The solution is optimal.

This means we did not have enough P-bins and/or F-bins to pack excess $MaxColor$ items and therefore cannot reduced D to 0. S contains at least one M-bins. However, because all the bins are topped by $MaxColor$ items, we cannot combine any bins to reduce the total number of bins. S is optimal.

4 Time Complexity

We analyze the time complexity based on the various cases. If D is less than or equal to 0, the algorithm first calls SPLIT which runs ZERO-WEIGHT to order the items based on color, then packs the items based on weight, using at $O(n)$ time in total where n is the number of items.

If D is more than 0, we consider two sub-cases: L is odd or L is even. If L is odd, we check if we can eventually reduce the D to 0. If we can, we pack items until D is 0. This takes at most $O(n)$, since we consider each item only once. When D is 0 we call SPLIT which takes $O(n)$ time. If we cannot reduce discrepancy to 0, we pack items by alternating colors. Since each item is considered exactly once, this takes $O(n)$ time.

If L is even, again we pack by alternating colors then call CONDENSE. The runtime of CONDENSE depends on the number of bins and since there will be $O(n)$ bins, CONDENSE runs in $O(n)$ time.

5 Bin Coloring Problem with 0-weight items of more than 2 colors, reordering allowed

In this section, we consider a version of the Bin Coloring Problem where items are of more than 2 colors, all items have a weight of 0, and reordering is allowed. More formally, the problem is defined as follows:

- The input to the problem is a string. Each character x represents an item with the color x .
- All items have a weight of 0.
- No adjacent items in the bin can be of the same color.
- Reordering is allowed. For example, for i and j such that $i > j$, the i th item in the string can be packed either before or after the j th item in the string.

We provide an algorithm ZERO-WEIGHT that is optimal for this problem.

Since the only constraint in the 0-weight situation is the color constraint (that no two adjacent items can have the same color), we pay special attention to the most frequent color, i.e. the color with the most number of items. In the first case, if this color has no more items than all the other colors combined, we can pack all the items in one bin, using other color items as buffers between items of the most frequent color. Specifically, we first alternate between items of other colors until we have exactly enough items of other colors to buffer the items of the most frequent color. Then, we alternate between items of other colors and items of the most frequent color. If on the other hand, the most frequent color occurs more than all the other colors combined, we are forced to use all the other color items as buffers. Even so, we will be running out of the other color items in the end and left with several items of the most frequent because there are too many of them. For each of the items of the most frequent color that are remaining, we are forced to open one new bin, since they are all of the same color.

Proof. First note that from a previous paper (Bin Packing with colors), if there are only two colors, then the minimum number of bins necessary is the discrepancy of the two colors. For

example, if there are 5 Black items and 3 White items, the number of bins necessary is $5 - 3 = 2$ bins. This is done by starting with an item of the color with more items (B) and alternating with items of the other color (W) until W runs out. We then add one more item of color B into this first bin. If $|B|$ and $|W|$ denote the number of black and white items, respectively, we need $|B| - |W| - 1$ more bins to fit the rest of the items of color W for a total of $|B| - |W|$ bins.

For more than 2 colors, we consider two cases:

Case 1: $\text{Discrepancy} \leq 0$: See below for an example. In this case there are at least as many OtherColors items as there are MaxColor items. In this case, we need only one bin, which is the fewest number of bins any algorithm can get. To do this, we alternate between items of OtherColors until the sum of the unused OtherColors items is one less than the number of MaxColors items. We then add one MaxColor item to the bin. From here, we alternate between any OtherColors item and a MaxColor item until all items are used. The last item will be one of MaxColor .

Note that it is always possible to alternate the OtherColors items until OtherCount is one less than the number of MaxColor items. The only case when we can no longer alternate is when there are items of only one color, X , remaining, and because of the color restriction, we can no longer put these items into the bin. We claim that at this point, OtherCount , which in this case is the number of X items, will be one less than MaxCount . We prove this by way of contradiction. Suppose when we stop alternating, OtherCount is equal or greater than MaxCount . Note that an X item must be on top of the bin, otherwise we have not completed the alternating step since we can pack another X item on top of the bin. There must be at least MaxCount X items remaining, so there were at least $\text{MaxCount} + 1$ total X items, which contradicts our assumption that MaxColor is the color with the maximum number of items.

After these alternations, we have one more MaxColor items than all OtherColors items, and it is easy to see that we can alternate between them and put all items in the same bin.

Case 2: $\text{Discrepancy} > 0$. In this case, all other items can be regarded as some uniform color non-MaxColor that is not equal to MaxColor , which makes this the black/white bin packing problem. This is because there are so many MaxColor items that we must pack each non-MaxColor item between two MaxColor items to use as few bins as possible. The first bin B_0 will start with a non-MaxColor item, alternate MaxColor and non-MaxColor items, and end with a non-MaxColor item, so will have $\text{OtherCount} * 2 + 1$ items in total. Each additional bin will have an item of MaxColor . Then the minimum number of bins necessary is still the difference between the number of MaxColor and non-MaxColor items, i.e. the discrepancy, as is shown in the Black/White bin packing problem.

□

The first step of the algorithm requires a scan of the input string to determine OtherCount and MaxCount . The second step requires a scan to separate the items into in two sets: items of

OtherColors and items of *MaxColor*. Packing each item takes constant time, so the remainder of the algorithm runs linear to the number of items. Therefore the algorithm has total runtime $O(3n)$ where n is the number of items.

EXAMPLE:

Items: WWWWWWWWWBYY (8W, 2B, 2Y)

In this case, *MaxColor* is W . *MaxCount* equals 8 and *OtherCount* equals 4. *Discrepancy* > 0.

According to our algorithm, we first create a bin B_0 of size $4 * 2 + 1 = 9$. Then we alternate between W and any *OtherColor* to fill the first bin.

WBWBWYWYW

Now, we have used all the *OtherColors*, and we still have 3 W 's remaining. Therefore, we create 3 new bins, each holding one of the remaining W 's.

Therefore, we get the following final result:

WBWBWYWYW / W / W / W

Algorithm 1 Algorithm UNITWEIGHT (I). Input is a string I of length n , representing items of different colors; L is the weight limit of each bin

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1: Scan  $I$  to count the number of times each color occurs. Also keep track of the color that
   occurs the most; let  $MaxColor$  denote this color and let  $MaxCount$  denote the number of
   items of  $MaxColor$ 
2: Let  $OtherColors$  denote the set of colors that are not  $MaxColor$  and let  $OtherCount$  denote
   the number of items of any color in  $OtherColors$ .
3: Let  $D = MaxCount - OtherCount$  //Discrepancy
4:  $S$  : solution //set of bins; initially empty
5: if  $D \leq 0$  then
6:   SPLIT( $I, n, L$ ) //SPLIT() gets solution from ZERO-WEIGHT() and splits it into bins of
   capacity  $L$ 
7:   Add the solution from SPLIT to  $S$ .
8:   return  $S$ ;
9: else
10:  if  $L$  is even then
11:    Create new current bin  $C$ . Start with placing a  $MaxColor$  item, alternating with an
         $OtherColors$  item, until  $C$  is full. Add  $C$  to  $S$ .
12:    Repeat the process until we run out of  $OtherColors$  items. Make sure any partially full
        bins are topped with a  $MaxColor$  item. There will be at most one such bin.
13:    Place the remaining  $MaxColor$  items into bins of their own. Add these bins to  $S$ .
14:    Call CONDENSE( $S$ ) to condense all the single-item bins. Add these bins to  $S$ .
15:  else if  $L$  is odd then
16:    if  $D \leq \lceil \frac{OTHER}{L/2} \rceil$  then
17:      //if there are enough  $OtherColors$  items such that alternating will allow the discrep-
        ance to be reduced to zero
18:      for 1 to  $D$  do
19:        Create new current bin  $C$ . Start with placing a  $MaxColor$  item, alternating with
        an  $OtherColors$  item, until  $C$  is full. Add  $C$  to  $S$ .
20:      end for
21:      Call SPLIT( $R$ ), where  $R$  is the remaining items to be placed. // $D=0$  now so SPLIT
        may be called
22:      Add the solution from SPLIT to  $S$ .
23:    else
24:      Create new current bin  $C$ . Start with placing a  $MaxColor$  item, alternating with an
         $OtherColors$  item, until  $C$  is full. Add  $C$  to  $S$ .
25:      Repeat this until all  $OtherColors$  items are used. Make sure any partially full bins
        are topped with a  $MaxColor$  item. There will be at most one such bin.
26:      Place the remaining  $MaxColor$  items in bins of their own. Add these to  $S$ .
27:    end if
28:  end if
29: end if
```

Algorithm 2 Algorithm SPLIT (I, n, L)

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1:  $R = \text{ZERO-WEIGHT}(I)$ :  
2: for  $i = 1$  to  $\left\lceil \frac{n}{L} \right\rceil$  do  
3:   Create new bin  $C$   
4:   Remove the first  $L$  items (or remaining items if there are fewer than  $L$ ) from  $R$  and pack  
   into  $C$   
5:   Add  $C$  to  $S$   
6: end for
```

Algorithm 3 Algorithm CONDENSE (S). S is a set of bins, both completely and partially filled

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1: Separate  $S$  into three sets respectively called M-bins, P-bins and F-bins.  
2: M-bins (MaxColor bins): contain only one item and the item is of MaxColor.  
3: P-bins (partially full bins): contain both MaxColor and OtherColors items, are topped  
   with a MaxColor item, and can fit at least two more items. There will be at most one of  
   this bin type.  
4: F-bins (full bins): fully packed bins that are topped with an OtherColors item.  
5: //Note that since CONDENSE is called only when  $L$  is even, all fully packed bins (F-bins) will  
   be topped with an OtherColors item, therefore all bins fall into one of these three categories.  
6: if P-bins is not empty then  
7:   Let current be the bin in P-bins  
8:   Remove current from P-bins  
9: else if M-bins is not empty then  
10:  Let current be a bin in M-bins  
11:  Remove current from M-bins  
12: end if  
13: while F-bins is not empty and M-bins is not empty do  
14:   if weight of current +2 >  $L$  then  
15:     Remove current from its containing set  
16:     Let current be the next bin in M-bins //switch to a new bin since current is full  
17:   end if  
18:   Let  $x$  be the top-most OtherColors item from a bin in F-bins  
19:   Let  $y$  be a MaxColor item from a bin (or the bin??) in M-bins  
20:   Pack  $x$  followed by  $y$  into current.  
21:   Remove  $x$  and  $y$  from their containing bins.  
22: end while
```

Algorithm 4 Algorithm ZERO-WEIGHT. Input is a string I , representing items of different colors.

- 1: Scan I to count the number of times each color occurs. Also keep track of the color that occurs the most; let $MaxColor$ denote this color and let $MaxCount$ denote the number of items of $MaxColor$
- 2: Let $OtherColors$ denote the set of colors that are not $MaxColor$ and let $OtherCount$ denote the number of items of any color in $OtherColors$.
- 3: $Discrepancy = MaxCount - OtherCount$
- 4: **if** $Discrepancy \leq 0$ **then**
- 5: Create a new bin B whose size is equal to the length of I .
- 6: Alternate between an item of $MaxColor$ and an item of any arbitrary $OtherColors$ until B is full and all items are packed. Only one bin is needed in this case.
- 7: **return** B
- 8: **else**
- 9: Create an empty set S that will be the solution
- 10: Create a bin B_0 of size $OtherCount * 2 + 1$. Add B_0 to S .
- 11: Alternate between an item of $MaxColor$ and an item of any arbitrary $OtherColors$ until B_0 is full. When B_0 is full, there will be no items of $OtherColors$ left.
- 12: Create a bin B_x for each of the remaining x items of $MaxColor$ and pack each of these items in one of these bins. In this case, the total number of bins is $Discrepancy$. Add each B_x to S
- 13: **return** S
- 14: **end if**
